B. Sc-part-I (Sub) Set Theory 10-07-2020

Countability of rational

Q.1. The union of a countable family of countable sets is countable.

Proof: Let $\{A_n\}$ be a countable family of sets that each A_n is countable. Let us consider

$A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \dots\}$

$A_2 = \{a_{21}, a_{22}, a_{22}, a_{24}, a_{25} \dots\}$

$A_3 = \{a_{31}, a_{32}, a_{33}, a_{34}, a_{35}, \dots\}$

$A_4 = \{a_{41}, a_{42}, a_{43}, a_{44}, a_{45}, \dots\}$

$A_5 = \{a_{51}, a_{52}, a_{53}, a_{54}, a_{55}, \dots\}$

$\dots$

$A_n = \{a_{n1}, a_{n2}, a_{n3}, a_{n4}, a_{n5}, \dots\}$

$\dots$

Here a_{ij} stands for the jth element of the ith set. Now, if we define the height of the element a_{ij} be $i+j$. Then, the height of a_{11} is $1+1=2$ and this is the only element of height 2. Again the height of each of the elements a_{12}, a_{21} is $1+2=2+1=3$ and these are the only elements of height 3. Similarly, the height of each of the elements a_{21}, a_{22}, a_{13} is $3+1=2+2=1+3=4$ and these are the only elements of height 4. Similar observations can be made about other elements. Hence we can arrange all elements according to their heights:

$a_{11}, a_{12}, a_{21}, a_{22}, a_{13}, a_{41}, \dots$

leaving out any element that has all ready occurred and hence all elements will be counted out and consequently $\cup A_n$ is countable.

Proved.

B.Sc. part I (Sub) Set theory

Countability of

Q.2 The set of all positive rationals is countable.

Proof: Let the set of all positive rationals

$$Q^+ = \{x : x = \frac{p}{q},\ p, q \in N\}$$

and arrange it in the following form

$$\begin{array}{lcccccccc}
A_1 & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & \cdots \\
A_2 & \frac{2}{1} & \boxed{\frac{2}{2}} & \frac{2}{3} & \boxed{\frac{2}{4}} & \frac{2}{5} & \boxed{\frac{2}{6}} & \cdots & \cdots \\
A_3 & \frac{3}{1} & \frac{3}{2} & \boxed{\frac{3}{3}} & \frac{3}{4} & \frac{3}{5} & \boxed{\frac{3}{6}} & \cdots & \cdots \\
A_4 & \frac{4}{1} & \boxed{\frac{4}{2}} & \frac{4}{3} & \boxed{\frac{4}{4}} & \frac{4}{5} & \boxed{\frac{4}{6}} & \cdots & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Here, A_n stands for all rationals with n as numerator. This table contains all positive rationals. Now arranging them in the order shown by the arrowed line in the above table avoiding repetition; we have,

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \cdots$$

which are to be counted as

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, \cdots$$

Thus the set of positive rationals is countable.